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VERY LONG CHAINS OF ANNIHILATOR IDEALS

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ABSTRACT
Here we examine one of the two right Goldie conditions: the ascending chain condition on right annihilator ideals, r. $ACC^{\perp}$. One might well ask if a ring has r. $ACC^{\perp}$, does there exist a bound on the lengths of chains of right annihilator ideals? Under certain additional hypotheses, this bound does exist. In general, however, a bound does not exist, as is shown by the two examples presented here.

Introduction

Here we examine one of the two right Goldie conditions: the ascending chain condition on right annihilator ideals, r. $ACC^{\perp}$. It is well known that if a ring has the other right Goldie condition, no infinite direct sums of right ideals, then there exists a bound on the lengths of the direct sums ([1], p. 8). One might well ask if a ring has r. $ACC^{\perp}$, does there exist a similar bound on the lengths of chains of right annihilator ideals? Under certain additional hypotheses, this bound does exist. For example, if the ring is semiprime, has r. $ACC^{\perp}$, then the bound exists if any one of the following holds:

(1) the ring satisfies a polynomial identity, a p.i. ([4], p. 17);

(2) the ring is affine with subexponential growth (also due to Small, unpublished);

(3) the ring is a right Goldie ring.

The third condition follows from Goldie's theorem which characterizes orders in semisimple right Artinian rings as semiprime right Goldie rings ([1], p. 23). A prime ring with a bound, n, on the indices of nilpotent elements, not only must satisfy $ACC^{\perp}$ (on both sides), but in addition has n as a bound on chains of 1-sided annihilator ideals ([3], p. 56). This extends the already known special

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case for $n = 1$, i.e. a prime ring with no nilpotent elements has no zero divisors. In general, however, a bound need not exist, even if we require the ring to have ACC on right and left annihilator ideals. A highly nonaffine, noncommutative example due to Fisher appeared in 1971 ([2], p. 516).

As evidenced by the above positive results, apparently there is a connection between the existence of a bound and the satisfying of other finiteness conditions. In the first part of this paper we present a commutative ring with $\mathrm{ACC}^{\perp}$ which has no bound. As in Fisher's example, this ring fails to be finitely generated. In addition, this ring contains a subring with the same properties, which is nilpotent. Hence this subring is as far from prime as possible. Then we construct a second ring which is noncommutative, finitely generated and prime with r. $\mathrm{ACC}^{\perp}$, but with no bound. In both examples the rings are isomorphic to their opposite rings, so all conditions are satisfied on both sides. These examples together with an example due to Irving of an affine p.i. ring with polynomially bounded growth, r. $\mathrm{ACC}^{\perp}$ but with no bound, show that the previously mentioned positive results cannot be improved.

A commutative ring with $\mathrm{ACC}^{\perp}$ and no bound

Let K be an arbitrary commutative domain with an infinite set of commuting indeterminates $(\bar{X}_{ij} \mid j \leqq i, i, j \in \mathbf{Z}^{+}\}$. Define I to be the homogeneous ideal generated by $\{\bar{X}_{ij}\bar{X}_{kl}\bar{X}_{nm}\}$ and $\{\bar{X}_{ij}\bar{X}_{ik}$, where $j \neq k\}$ in the polynomial ring $P = K[\{\bar{X}_{ij}\}]$. The ring we want is the factor ring $R = P/I$. Let x_{ij} be the image of $\bar{X}_{ij}$ in R, and let $X = \{x_{ij}\}$. To clarify the relations among the x_{ij} one might examine the following infinite array:

$$\begin{array}{lll} x_{11} & & \\ x_{21} & x_{22} & \\ x_{31} & x_{32} & x_{33} \\ \vdots & & \end{array}$$

The product of any two distinct elements in the same row is zero. The product of any three elements in the array is zero.

We have a natural grading on R by letting the elements in X have degree 1, elements in K have degree 0. Then $R = R_0 \oplus R_1 \oplus R_2$, where $R_i = \{r \in R \mid r$ is homogeneous of degree $i\}$. The set $\{x_{ij}\}$ is a basis over K for R_1, $\{x_{ij}x_{kl} \mid (i \neq k)$ or $(i = k$ and $j = l)\}$ is a basis over K for R_2. Note that $(R_1 R)^3 = 0$. We will see that the ideal $R_1 R$ contains all the zero divisors in R. Let $r(S)$ denote the right annihilator of S.

PROPOSITION 1. *R has ascending chains of annihilator ideals of arbitrary lengths.*

PROOF. For $n \in \mathbf{Z}^{+}$, consider:

$$A_{n1} = r\{x_{nj}, 1 \leqq j \leqq n\} = R_2$$
$$\vdots$$
$$A_{nk} = r\{x_{nj}, k \leqq j \leqq n\} = \left(\sum_{l=1}^{k-1} x_{nl}R\right) + R_2$$
$$\vdots$$
$$A_{nn} = r\{x_{nn}\} = \left(\sum_{l=1}^{n-1} x_{nl}R\right) + R_2$$

Then

$$A_{n1} \subsetneqq A_{n2} \subsetneqq \cdots \subsetneqq A_{nn} \text{ is a chain of length } n-1.$$

Now we show that R has ACC$^{\perp}$. We begin by explicitly determining the form of all annihilator ideals.

LEMMA 1. *For each positive integer n, let $X_n = \{x_{nj}, 1 \leqq j \leqq n\}$. Let S be a subset of R.* (i) *If $S \subset R_2$, then $r(S) = R_1R$.* (ii) *$S \not\subset R_1R$, then $r(S) = (0)$.* (iii) *If $S \subset R_1R$ and $S \not\subset R_2$, then either $r(S) = R_2$, or there exists n such that $Y \subsetneqq X_n$, and $r(S) = YR + R_2$.*

PROOF. (i) If $S \subset R_2$, then obviously $r(S) = R_1R$.

(ii) First we examine when the product of two elements can be zero. Let $p, q \in R$, and let p_i, q_i be the components of p, q in R_i. Suppose $pq = 0$. That is, all the components of the product pq are zero:

$$p_0q_0 = 0, \tag{1}$$

$$p_0q_1 + p_1q_0 = 0, \tag{2}$$

$$p_0q_2 + p_1q_1 + p_2q_0 = 0. \tag{3}$$

Assume $p \notin R_1R$; that is, $p_0 \neq 0$. Because R is a torsion-free K-module, we obtain successively from equations (1), (2), and (3) that $q_0 = 0$, $q_1 = 0$, and $q_2 = 0$. Thus, if $S \not\subset R_1R$, then $r(S) = (0)$. In particular, all zero divisors in R lie in R_1R.

(iii) Now assume $p, q \in R_1R$. Then $pq = p_1q_1$ and p_1 has the form $\Sigma a_{ij}x_{ij}$, q_1 has the form $\Sigma b_{ij}x_{ij}$, for some a_{ij}, $b_{ij} \in K$. So

$$p_1q_1 = \sum_{i,j} a_{ij}b_{ij}x_{ij}^2 + \sum_{j<k}(a_{ij}b_{ik} + a_{ik}b_{ij})x_{ij}x_{ik} + \sum_{i<k}(a_{ij}b_{kl} + a_{kl}b_{ij})x_{ij}x_{kl}.$$

Recall that $x_{ij}x_{ik} = 0$ for $j \neq k$. Hence

$$p_1 q_1 = \sum_{i,j} a_{ij} b_{ij} x_{ij}^2 + \sum_{i<k} (a_{ij} b_{kl} + a_{kl} b_{ij}) x_{ij} x_{kl}.$$

Thus $pq = 0$ iff both of the following hold:

$$a_{ij} b_{ij} = 0, \tag{4}$$

$$a_{ij} b_{kl} + a_{kl} b_{ij} = 0, \qquad \text{for all } i \neq k. \tag{5}$$

Next we determine $r(p)$ for $p \in R_1 R$, $p \notin R_2$. Because $p \notin R_2$, p_1 has the form $\Sigma a_{ij} x_{ij}$ such that at least one a_{ij} is nonzero. For the remainder of this proof assume $a_{nm} \neq 0$. Let $q \in r(p)$. Then $q_0 = 0$ and q_1 has the form $\Sigma b_{ij} x_{ij}$ where $b_{nm} = 0$ from (4). Then (5) implies $a_{ij} b_{nm} + a_{nm} b_{ij} = a_{nm} b_{ij} = 0$ for all $i \neq n$. Consequently, $b_{ij} = 0$ for all $i \neq n$. Thus q_1 has the form $\Sigma_{l \neq m} b_{nl} x_{nl}$. Symmetrically, if one of the b_{nl}'s is nonzero, then p_1 must have the form $\Sigma a_{nj} x_{nj}$. That is, if there exists $k \neq n$ such that $a_{kj} \neq 0$, then all the b_{nl}'s must be zero. Thus, in this case, $r(p) = R_2$. On the other hand, if $p_1 = \Sigma a_{nj} x_{nj}$ then $r(p) = YR + R_2$, $Y = \{x_{nj} \mid a_{nj} = 0\}$.

Now consider $S \subset R_1 R$, $S \not\subset R_2$. Note that $(X_k R + R_2) \cap (X_l R + R_2) = R_2$ for $k \neq l$. Because $r(S) = \cap_{p \in S} r(p)$, for some n there exists $Y \subset X_n$, possibly the empty set, such that $r(S)$ has the form $YR + R_2$. This completes the proof of Lemma 1.

PROPOSITION 2. *R has* $\mathrm{ACC}^{\perp}$.

PROOF. Suppose A is a nontrivial annihilator ideal other than $R_1 R$. Then Lemma 1 implies there exists $n \in \mathbf{Z}^+$, and $Y \subset X_n$ such that A has the form $YR + R_2$. Note that X_n has exactly $2^n - 1$ proper subsets. So A contains only finitely many annihilator ideals. Thus R has the descending chain condition on annihilators. Because R is commutative, we conclude that R has $\mathrm{ACC}^{\perp}$.

By considering $R_1 R$ instead of R_1 we have a nilpotent commutative ring with $\mathrm{ACC}^{\perp}$, and with no bound on the length of its chains of annihilator ideals.

An affine prime ring with $\mathrm{ACC}^{\perp}$ and no bound

This example is much more intricate than the commutative one. We begin with $K = \mathbf{Z}/2\mathbf{Z}$ (actually we could use an arbitrary domain) and a set of four noncommuting indeterminates $\{X, Y, Z, W\}$. We consider the graded factor ring, $R = A/I$, where A is the free algebra $K\{X, Y, Z, W\}$ and I is the homogeneous two-sided ideal of A generated by the following set of monomials:

$$B = \{ZY^i WX^k WY^i Z, \, i \geqq 0, 0 \leqq k \leqq i\}.$$

R is given the grading induced from the standard grading (by total degree) on A, in which X, Y, Z, W each have degree one. Note also that the canonical involution on A given by reversing the order of the "letters" in each monomial "word" fixes each element of B. Hence there is an induced involution on R. Let x, y, z, w be the images of X, Y, Z, W respectively. Throughout this example greek letters will always denote monomials in x, y, z, w. Certain monomial zero divisors, those which are images of the divisors of elements in B, play a key role in all that follows. Let U be the set of images of left divisors of elements in B and let V be the set of images of right divisors of elements in B. Note that the antiautomorphism of R maps U onto V (and vice versa). The letters μ and ν will always represent elements of U and V respectively. Denote by $\alpha \mid_l \beta$ (resp. $\alpha \mid_r \beta$) that α divides β on the left (resp. right).

Our strategy is to explicitly determine the right annihilator ideals by first examining the right annihilators of monomials, and then the right annihilators of general elements in R. From this we obtain the form of a right annihilator ideal. Then it is not difficult to prove that R is prime and has r. ACC$^\perp$.

Note that the ideal I of A has a K-base of monomials, so the right annihilator of a monomial in R, as well as the intersection of such annihilators, is generated by monomials. Consider two nonzero monomials α, β whose product is zero. Then some $zy^iwx^kwy^iz$ must occur in the product $\alpha\beta$, but not in α nor in β. Thus α, β must have the forms $\alpha'\mu$, $\nu\beta'$ with $\mu \in U$, $\nu \in V$. Because μ is uniquely determined from α we have $r(\alpha) = r(\mu)$. Similarly $l(\beta) = l(\nu)$. So we need only consider monomials in U and V. The right annihilators of elements in U are given by:

(a) For any $t \geqq 0$,

$$r(zy^t) = \sum_{\nu \in V_{2,t}} \nu R,$$

$$V_{2,t} = \{y^{i-t}wx^kwy^iz, 0 \leqq k \leqq i, i \geqq t\}.$$

(b) For any $0 \leqq t \leqq i$, $i \geqq 0$,

$$r(zy^iwx^t) = \sum_{k=0}^{i-t} x^kwy^izR.$$

(c) For any i, k, t with $0 \leqq k \leqq i$, $0 \leqq t \leqq i$, $i \geqq 0$.

$$r(zy^iwx^kwy^t) = y^{i-t}zR.$$

We obtain dual expressions for the left annihilators of elements in V.

Next, we make several observations which will be heavily used in the proofs of Propositions 5 and 6.

OBSERVATIONS. (1) Note that in each case, $r(\mu)$ is generated by $r(\mu)\cap V$. Let $U_i=\{\mu\in U \mid \deg_w \mu = i\}$, $V_i=\{\nu\in V\mid \deg_w \nu = i\}$. For $\mu\in U_0$, $r(\mu)\cap V$ is an infinite subset of V_2 (case (a) above). If $\mu\in U_2$, then $r(\mu)\cap V$ has exactly one element (case (c)). The analogous results are true for elements of V_i.

(2) For any $\nu\in V_{2,t}$ we have $l(\nu)=\mathrm{R}zy^t$, by the dual of case (c). Hence if $\mu\in U_0$ and $\nu\in r(\mu)\cap V$, then $l(\nu)=R\mu$.

(3) Suppose $r(\mu)\cap r(\mu')\neq 0$. Then $r(\mu)\cap r(\mu')\cap V\neq 0$. Note $\deg_w\mu=\deg_w\mu'$ because if $\mu\in U_i$ then $r(\mu)\cap V\subseteq V_{2-i}$.

(i) If $\mu\in U_0$, then $\mu=\mu'$. Thus $r(\mu)=r(\mu')$.

(ii) If $\mu\in U_2$, then from case (c) above, $r(\mu)=r(\mu')$.

(iii) If $\mu\in U_1$ then the i for which $zy^iw\,|_l\mu$ is determined by any $\nu\in r(\mu)\cap V$. Thus $zy^iw\,|_l\mu'$ and, so, $wy^iz\,|_r\nu'$ for all $\nu'\in r(\mu')\cap V$. In other words, $r(\mu')\cap V$ lies in a finite subset of V_1 completely determined by any single element in $r(\mu)\cap V$.

PROPOSITION 3. *R has chains of right (resp. left) annihilator ideals of arbitrary lengths.*

PROOF. Let $A_{t,i}=r(zy^iwx^t)=\sum_{k=0}^{i-t}x^kwy^izR$, for $0\leqq t\leqq i$. Then $A_{i,i}\subsetneqq\cdots\subsetneqq A_{0,i}$. An application of the involution gives corresponding chains of left annihilators.

Now we can consider the right annihilator of the general element in R. Let C be the set of all elements of R of the form $1+\sum_{\text{finite}}\alpha_i$, where the α_i are monomials. (We assume $1\in C$, corresponding to the trivial sum.) For the rest of the paper the letters c and d will always denote elements of C. Take any $r,s\in R$. Then clearly there exist elements α_i,β_j,c_i,d_j such that $r=\sum_{i=1}^n\alpha_ic_i$, $s=\sum_{l=1}^m d_l\beta_l$, and for each $j<k$, all of the following hold: $\deg\alpha_j\leqq\deg\alpha_k$, $\deg\beta_j\leqq\deg\beta_k$, $\alpha_j\nmid_l\alpha_k$, and $\beta_j\nmid_r\beta_k$. (Simply group the monomials in r (resp. s) according to greatest common left (resp. right) divisors.)

PROPOSITION 4. *Using the above terminology, $rs=0$ iff $\alpha_ic_id_j\beta_j=0$ for all i,j. Moreover, if $\alpha cd\beta=0$ then α,β have the forms $\alpha'\mu$, $\nu\beta'$ where $\mu\nu=0$ and $\mu cd\nu=0$.*

Observe that Proposition 4 implies that the right annihilator of an arbitrary element is a finite intersection of right annihilators of elements of the form μc. Furthermore, elements of the form $d\nu$ generate $r(\mu c)$.

PROOF OF PROPOSITION 4. Suppose $rs=0$ but $\alpha_ic_is\neq 0$ for some i. Each nonzero monomial $\alpha_i\gamma$ in α_ic_is must be cancelled by some other term in the monomial expansion of rs. Hence, $\alpha_i\gamma = \alpha_j\delta$ for some $j \neq i$. But then $\alpha_i \mid_l \alpha_j$, or vice versa, contradicting the conditions on the α_i's. Hence $\alpha_i c_i s = 0$ for all i. A symmetric argument yields $\alpha_i c_i d_j \beta_j = 0$ for all i, j.

Now suppose $0 = \alpha c d \beta$. From the form of c and d, we have $\alpha c d \beta = \alpha\beta + \Sigma$(higher degree monomials). Hence, $\alpha\beta = 0$. By the earlier consideration of monomial annihilators, α and β must have the forms $\alpha'\mu$ and $\nu\beta'$ respectively, with $r(\alpha) = r(\mu)$ and $l(\beta)$ and $l(\nu)$. Hence $\alpha c d \beta = 0$ iff $\mu c d \nu = 0$.

PROPOSITION 5. *If $S \subset R$ such that $r(S) \neq 0$, then $r(S)$ has one of the following forms*: (1) $\Sigma_{\nu\in V_{2,t}} d\nu R$, *for some* $t \geqq 0$, $d \in C$ *or* (2) $\Sigma_{i=1}^{n} d_i \nu_i R$, *for some* $d_i \in C$.

PROOF. Without loss of generality, $S \subset \{\mu_j c_j\}$. Define $U_S = \{\mu \in U \mid \mu c \in S$ for some $c \in C\}$, $V_{r(S)} = \{\nu \in V \mid d\nu \in r(S)$ for some $d \in C\}$, $D_{r(S)} = \{d \in R \mid d\nu \in r(S)$ for some $\nu \in V\}$. Proposition 4 implies $U_S V_{r(S)} = 0$.

Assume $V_{r(S)}$ is infinite. From observations (1) and (3) above $U_S = \{zy^t\}$, for some t. Thus $V_{r(S)} \subset V_{2,t}$.

CLAIM 1. $V_{r(S)} = V_{2,t}$. Moreover, if $d \in D_{r(S)}$, then, for every $\nu \in V_{2,t}$, $d\nu \in r(S)$.

PROOF OF CLAIM 1. Take any $d \in D_{r(S)}$ and any $\nu' \in V_{2,t}$ such that $Sd\nu' = 0$. By observation (2), for each $\nu \in V_{2,t}$ we have $l(\nu) = Rzy^t$. Hence $(Sd)\nu' = 0$ implies $(Sd)\nu \subset (Rzy^t)\nu = 0$, for every $\nu \in V_{2,t}$, as claimed.

CLAIM 2. If d is any element in $D_{r(S)}$ of minimal degree, then $r(S) = \Sigma_{\nu\in V_{2,t}} d\nu R$.

PROOF OF CLAIM 2. If not, pick $d' \in D_{r(S)}$ of minimal degree, such that, for some $\nu' \in V_{2,t}$, $d'\nu' \notin \Sigma_{\nu\in V_{2,t}} d\nu R$. Of course, $\deg d' \geqq \deg d$. Let $e = (d - d')\nu' \in r(S)$. Since d, $d' \in C$, $d - d'$ has no constant term. Hence every monomial of e has degree strictly larger than $\deg \nu'$. Write $e = \Sigma_{l=1}^{m} d_l \beta_l$ where for each $j < k$ we have $\deg \beta_j \leqq \deg \beta_k$ and $\beta_j \nmid_r \beta_k$. These conditions assure that no nonzero monomial occurs in more than one $d_l\beta_l$. By Proposition 4, each β_l has the form $\nu_l\delta_l$, where $\nu_l \in r(S) \cap V = V_{2,t}$ and $d_l\nu_l \in r(S)$. Since $d_l \in C$, β_l occurs as a monomial of $d_l\beta_l$, hence of e. So, $\deg \beta_l > \deg \nu'$. But, if τ_l is a monomial of highest degree in d_l, then $\tau_l\beta_l$ also occurs in $d_l\beta_l$, so in e. Hence,

$$\deg d_l + \deg \beta_l = \deg \tau_l\beta_l \leqq \deg e \leqq \deg(d - d') + \deg \nu' \leqq \deg d' + \deg \nu'.$$

Because $\deg \beta_l > \deg \nu'$, we conclude $\deg d_l < \deg d'$, for each l. It follows by the choice of d' that each $d_l\nu_l \in \Sigma_{\nu\in V_{2,t}} d\nu R$. Hence $d'\nu' = d\nu' - e \in \Sigma_{\nu\in V_{2,t}} d\nu R$, a contradiction.

Now assume $V_{r(S)}$ is finite. Then clearly, from the proof of Claim 1, we have $V_{r(S)} \cap V_2 = \emptyset$. For each $\nu_i \in V_{r(S)}$, choose d_i of minimal degree such that $d_i \nu_i \in r(S)$. Then apply the argument used for Claim 2 to obtain $r(S) = \sum_{\nu_i \in V_{r(S)}} d_i \nu_i R$.

REMARK. Suppose $r(S) = \sum d_i \nu_i R$, where the d_i's are of minimal degree as above. If $d'_i \nu_i \in r(S)$ and $\deg d'_i = \deg d_i$, then $r(S) = \sum d'_i \nu_i R$.

PROPOSITION 6. *R has* r. $\mathrm{ACC}^{\perp}$.

PROOF. Suppose $r(S_1) \subset r(S_2)$. Clearly $V_{r(S_1)} \subseteq V_{r(S_2)}$. If $V_{r(S_1)}$ is infinite, then, by observation (3), and Proposition 5, $V_{r(S_1)} = V_{r(S_2)} = V_{2,t}$ for some t. So there exist d_1, d_2 as in Proposition 5 such that

$$\sum_{\nu \in V_{2,t}} d_1 \nu R = r(S_1) \subset r(S_2) = \sum_{\nu \in V_{2,t}} d_2 \nu R.$$

Because $d_1 \nu \in r(S_2)$ and d_2 is of minimal degree in $D_{r(S_2)}$, we have $\deg d_1 \geqq \deg d_2$. By the above remark, if $\deg d_1 = \deg d_2$, then $r(S_1) = r(S_2)$. Thus any chain of right annihilators containing $r(S_1)$ may have at most $\deg d_1 + 1$ elements.

Suppose $V_{r(S_1)}$ is finite, $r(S_1) \subset r(S_2) \subset \cdots r(S_t) \subset \cdots$ and $r(S_j) = \sum_{\nu_i \in V_{r(S_j)}} d_{j,i} \nu_i R$. By observation (3) there is a finite set V' of V determined only by $r(S_1)$ such that, for all j, $V_{r(S_j)} \subset V'$. Now if $V_{r(S_j)} = V_{r(S_1)}$, then, as in the infinite case, $j \leqq \sum(\deg d_{1,i} + 1)$. Because V' is finite there are only finitely many j's for which $V_{r(S_j)} \subsetneqq V_{r(S_{j+1})}$. Therefore the chain must terminate.

PROPOSITION 7. *R is prime.*

PROOF. For any nonzero $s \in R$, $r(swyw) = 0$.

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